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LIMITS OF THE PRISMOIDAL FORMULA.

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CONSIDER the solid represeted by the following integral:

$$V=\int_{x=0}^{x=a}\int_{y=0}^{y=f(x)}\int_{z=0}^{z=\psi(x,y)}dx\,dy\,dz.\;.\;.\;.\;.\;.\;.\;.\qquad(1)$$

If the Prismoidal formula holds for this volume, we shall ascertain the fact by comparing the result of the integration of (1) with the result obtained by placing in the formula

$$V=\tfrac{a}{6}[\mathcal{A}_0+4\mathcal{A}_{\frac{1}{2}a}+\mathcal{A}_a]$$

(in which $\mathcal{A}_0$ is the section by the plane YZ, $\mathcal{A}_{\frac{1}{2}a}$ that by the plane $x=\frac{1}{2}a$ &c.) the values of the integral

$$\mathcal{A}=\int_{y=0}^{y=f(x)}\int_{z=0}^{z=\psi(x,y)}dy\,dz\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\qquad(2)$$

when x is made successively equal to 0, $\frac{1}{2}a$ and a.

It is evident in the first place that the functions $f(x)$ a d $\psi(x,y)$ must be such that, on the integration of (1) with referenec to x, no *new* function shall be introduced different from what existed previously, such as a logarithm; for as this would not appear in (2), the results obtained by integration, and by the formula could not agree. It will be seen also that in (1) the result of the first two integrations must not contain radicals involving x, since on integrating again these would be raised to higher powers, which would not be done by the formula.

Let us then assume $\psi(x,y)$ a rational integral expression of the form

$$z=\psi(x,y)=\left|\begin{array}{l}Ax^my^n+Bx^my^{n-1}+Cx^{m-1}y^n+\ldots\\+Ex^m+Fx^{m-1}+Gx^{m-2}+\ldots\\+Ky^n+L^{n-1}+My^{n-2}+\ldots\\+P\end{array}\right\}\;.\;.\;.\;.\qquad(3)$$

in which m and n are positive integers.

Since each term of this polynomial is to be integrated by itself we will consider only the term of highest degree Ax^my^n. Thus

$$V_1=A\int_0^a\int_0^{f(x)}x^my^ndx\,dy=A\int_0^a\left[\frac{x^my^{n+1}}{n+1}\right]_0^{f(x)}dx.\;.\;.\;.\qquad(4)$$

Let $\qquad y=f(x)=A_1x^p+B_1x^{p-1}+\ldots.+Q\;.\;.\;.\;.\;.\;.\;.\qquad(5)$

and as before let us consider only the first term A_1x^p; then

$$V_2=\frac{AA_1^{n+1}}{n+1}\int_0^a x^{p(n+1)+m}dx=\frac{AA_1^{n+1}}{n+1}\cdot\frac{a^{p(n+1)+m+1}}{p(n+1)+m+1}.\;.\;.\;.\qquad(6)$$

Now let us apply the Prismoidal Formula. From equation (2)

$$\boldsymbol{A}_0 = \int_{y=0}^{y=f(0)} \int_{z=0}^{z=\psi(0,y)} dy\,dz = \int_0^Q (Ky^n + Ly^{n-1} + \ldots + P)dx$$

$$= KQy^n + LQy^{n-1} + \ldots + PQ \ldots \ldots . (7)$$

If however we consider only the term integrated in (6) we have

$$\boldsymbol{A}_{0\,.\,2} = 0,$$

and similarly

$$\boldsymbol{A}_{\frac{1}{2}a\,.\,2} = \frac{AA_1^{n+1}}{n+1} \cdot \frac{a^{p(n+1)+m}}{2^{p(n+1)+m}},$$

$$\boldsymbol{A}_{a\,.\,2} = \frac{AA_1^{n+1}}{n+1} \cdot a^{p(n+1)+m} .$$

$$\therefore \quad V'_2 = \frac{a}{6}\left[0 + \frac{4AA_1^{n+1}}{n+1} \cdot \frac{a^{p(n+1)+m}}{2^{p(n+1)+m}} + \frac{AA_1^{n+1}}{n+1} \cdot a^{p(n+1)+m} \right]$$

$$= \frac{AA_1^{n+1}}{n+1}\left[\frac{a^{p(n+1)+m+1}}{3 \times 2^{p(n+1)+m-1}} + \frac{a^{p(n+1)+m+1}}{6} \right] \ldots (8)$$

The difference between the results obtained by the formula and by integration is

$$V'_2 - V_2 = \frac{AA_1^{n+1} a^{p(n+1)+m+1}}{n+1} \left[\frac{1}{3 \times 2^{p(n+1)+m-1}} + \frac{1}{6} - \frac{1}{p(n+1)+m+1} \right].$$

If the quantity in the parenthesis is zero the prismoidal formula must be correct. In the parenthesis let

$p(n+1) + m = 1$, then $\frac{1}{3} + \frac{1}{6} - \frac{1}{2} = 0$;
$p(n+1) + m = 2$, " $\frac{1}{6} + \frac{1}{6} - \frac{1}{3} = 0$;
$p(n+1) + m = 3$, " $\frac{1}{12} + \frac{1}{6} - \frac{1}{4} = 0$;
$p(n+1) + m = 4$, " $\frac{1}{24} + \frac{1}{6} - \frac{1}{5} = \frac{1}{120}$;
$p(n+1) + m = 5$, " $\frac{1}{48} + \frac{1}{6} - \frac{1}{6} = \frac{1}{48}$;
$p(n+1) + m = 6$, " $\frac{1}{96} + \frac{1}{6} - \frac{1}{7} = \frac{23}{672}$, &c.

Thus it appears that the formula is perfectly correct so long as $p(n+1) + m$ does not exceed *three*, and that from this point on there is a constantly increasing error.

Let us consider the cases that arise when different values are assigned to p.

1st $p = 3$, $\therefore m = n = 0$:
2nd $p = 2$, $\therefore m = 1, n = 0$:
3rd $p = 1$, $\therefore m = 0, n = 2$, or $m=1, n=1$, or $m=2, n=0$:
4th $p = 0$, $\therefore m = 3, n =$ any finite number.

In the first Case $f(x) = A_1x^3 + B_1x^2 + C_1x + Q = y$, and $\psi(x,y) = P = z$, thus the upper surface is a horizontal plane.

In the 2nd Case $f(x) = A_1x^2 + B_1x + Q = y$, and $\psi(x,y) = A'x + P' = z$, a plane perpendicular to ZX.

In the third Case $f(x) = A_1x + Q = y$,
and $\quad \phi(x, y) = A'x^2 + B'xy + C'y^2 + D'x + E'y + P = z$,
an elliptic or hyperbolic paraboloid.

In the fourth Case $f(x) = Q = y$, and $\phi(x, y)$ is a rational integral function of x and y in which y may appear with *any* exponent, but x must not be of higher degree than the 3rd. The surface represented by $\phi(x, y)$ may then be made, in this case, to pass through any number of points, provided that not more than four are in any single plane parallel to ZX.

Let us now determine the limits of Weddle's Formula for seven equidistant cross-sections,—

$$V = \frac{a}{20}\left[\boldsymbol{A}_0 + 5\boldsymbol{A}_{\frac{a}{6}} + \boldsymbol{A}_{2\frac{a}{6}} + 6\boldsymbol{A}_{3\frac{a}{6}} + \boldsymbol{A}_{4\frac{a}{6}} + 5\boldsymbol{A}_{5\frac{a}{6}} + \boldsymbol{A}_a\right] \ldots (13)$$

Proceeding precisely as before we find

$$\boldsymbol{A}_0 = 0,$$

$$5\boldsymbol{A}_{\frac{a}{6}} = \frac{5AA_1^{n+1}}{n+1} \cdot \frac{a^{p(n+1)+m}}{6^{p(n+1)+m}},$$

$$\boldsymbol{A}_{2\frac{a}{6}} = \frac{AA_1^{n+1}}{n+1} \cdot \frac{2^{p(n+1)+m}\,a^{p(n+1)+m}}{6^{p(n+1)+m}} \text{ \&c.; hence}$$

$$V''_2 = \frac{AA_1^{n+1}a^{p(n+1)+m+1}}{n+1} \times$$

$$\left[\frac{5 + 2^{p(n+1)+m} + 6 \times 3^{p(n+1)+m} + 4^{p(n+1)+m} + 5^{p(n+1)+m+1} + 6^{p(n+1)+m}}{20 \times 6^{p(n+1)+m}}\right] \ldots (14)$$

Subtract (6) from (14) and let the difference $= \dfrac{\delta AA_1^{n+1}\,a^{p(n+1)+m+1}}{n+1}$.

Make $p(n + 1) + m$ equal successively to 1, 2, 3. &c.

$$\therefore \quad \delta_1 = 0, \quad \delta_2 = 0, \quad \delta_3 = 0, \quad \delta_4 = 0, \quad \delta_5 = 0,$$

$$\delta_6 = \frac{1}{54432}, \quad \delta_7 = \frac{1}{15552}, \quad \delta_8 = \frac{113}{699840}. \left(= \frac{1}{6193} \text{ nearly.}\right)$$

Thus the formula is *exact* when $p(n + 1) + m$ does not exceed 5, and the error is small when $p(n + 1) + m$ is 6, or even 7.

The formula gives each term which is of too high degree a numerical value greater than that obtained by integration, but whether the final result is to be greater or less depends upon the signs of the coefficients of the terms of too high degree.

Giving different values to p, we have, when $p = 0$, $n =$ any number whatever, and $m = 5$ at most if the result is to be exact.

If $p = 1$, we have $f(x) = A_1x + Q$,

$$\begin{aligned} \text{and} \quad z = \phi(x, y) = Ax^4 &+ Bx^3y + Cx^2y^2 + Dxy^3 + Ey^4 \\ &+ Fx^3 + Gx^2y + Hxy^2 + Iy^3 \\ &+ Kx^2 + Lxy + My^2 + Nx + Oy + P. \end{aligned}$$

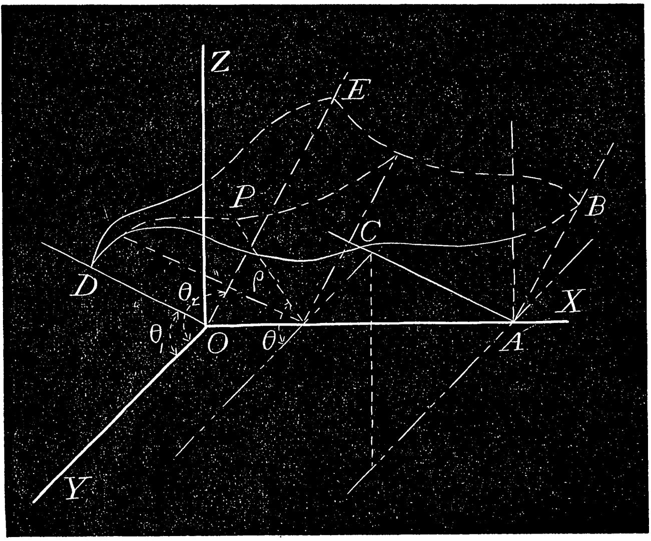
If $p = 2$, $\quad f(x) = A_1x^2 + B_1x + Q$, and

$$z = \psi(x, y) = Ax^3 + Bx^2 + Cxy + Dx + Ey + P.$$

If $p = 3$, $\quad f(x) = A_1x^3 + B_1x^2 + C_1x + Q$, and

$$z = \psi(x, y) = Ax^2 + Bx + P, \text{ a parabolic cylinder.}$$

If $p = 4$ the upper limiting surface becomes a plane parallel to Y, and if $p = 5$ it is a plane parallel to XY.



Let us now consider the wedge-shaped solid $OABCDE$, using $x\theta\rho$ as coordinates instead of xyz, as shown in the figure. The volume is then

$$V = \int_0^a\int_{\theta_1}^{\theta_2}\int_0^{\rho=\psi(x,\theta)} dx.\rho d\theta.d\rho = \frac{1}{2}\int_0^a\int_{\theta_1}^{\theta_2}\Big[\rho^2\Big]_0^{\psi_1(x,\theta)} dx\, d\theta \ldots (15)$$

$$\text{Let } \rho = \psi_1(x, \theta) = \sqrt{[x^m f_1'(\theta) + x^{m-1}f_2'(\theta) + \ldots]}, \ldots\ldots\ldots (16)$$

in which

$$f_1'(\theta) = \frac{d}{d\theta}\Big[f_1(\theta)\Big].$$

$\therefore$ considering as before only the first term we have

$$V_1 = \frac{1}{2}\int_0^a\int_{\theta_1}^{\theta_2} x^m f_1'(\theta)\, dx\, d\theta = \frac{1}{2}\Big[f_1(\theta_2) - f_1(\theta_1)\Big]\frac{a^{m+1}}{m+1}. \ldots (17)$$

Now both the Prismoidal Formula and that of Weddle hold for this case to the same extent as for that treated before, since we have only to make p equal 0 in (6) to reduce it to a form similar to (16).

Hence the prismoidal formula holds when x enters no term of $\psi(x, y)$ or $\psi_1(x, \theta)$ to a degree higher than the third, and Weddle's when x enters no term to a degree higher than the fifth; while y or θ may enter in any form algebraic or transcendental, so long as the second integration is between constant limits.

It is evident that these formulæ hold to the same extent for the areas of plane curves in ZX.

Weddle's formula seems decidedly preferable to the Prismoidal for accuracy either for curves or solids.